20

DYNAMICS OF VIBRATION OF A CANTILEVER UNDER LATERAL IMPACT OF AN ELASTIC LOAD

(General Theory—Part I)

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ABSTRACT. In this paper (which forms part I of a series of papers to appear), the writer works out the dynamics of vibration of a cantilever due to lateral impact of an elastic load at the free end as also at its middle The cantilever is at rest before impact begins and is supposed to behave like a loaded beam so long as the load is on contact with it The elastic load is taken to be hard load backed by weightless spring Each of the expressions for displacement of the bar and pressure of impact comes out in forms of respective series involving directly measureable quantities, mass, length, shape and young's modulus of the material of the cantilever and the striking distance as also the striking velocity and the mass of the load Unlike previous theories, the present theory is built up without assuming any law of force between the load and the beam The theory is perfectly general and can be easily followed in case of beams with different end conditions The agreement between the theory and the experiment is remarkable

INTRODUCTION

Young (1807), Hodgkonson (1833), Cox (1856), St. Venant (1883), Timoshenko (1956),, Hoppman (1948), Ghosh Ray (1955) and others have tried to develop the dynamics of vibration of a beam under lateral impact of a load, H. L Mason (1936), Hoppman, Timoshenko, have given a partial account of the history of beam-impact problems. The major analytical investigation of the problem started from Cox H., who assumed the deflection of the beam under dynamic condition of impact has the same value as given by statical deflection condition. Further he assumed that the impinging ball moves with the beam until the kinetic energy of the system is completely transformed into potential energy of bending St Venant (1883), developed a theory on the assumption that the impinging body remains attached with the beam for half the period of vibration. He suggested that the vibration of such a beam can be expressed as a series of normal functions. His case can be looked upon as a loaded beam excited by initial velocity at the point of loading Experimental observations, made by the writer as also other workers showing existence of multiple contacts within a total period of impact and the load terminating its contact before the maximum deflection is reached, give evidence contrary to these assumptions.

Timoshenko derived the integral equation for the case of the central impact of a load on a simply supported beam. He assumed a definite law of force between ball and the beam and worked out the problem in line of forced vibration. Further taking into account, the local deformation of the contact region, he considered Hertz's theory of impact which depends on a knowledge of the geometric and elastic properties of materials at contact surface in specific beam- load system to obtain expression for displacement of the beam Further, as remarked by Hoppman, the procedure consumes much time and so it is not satisfactory. But the solution of the problem as given by Hoppman, on the basis of an assumption of a 'normalised force' of sinusoidal character involve prior determination of some quantities, one of these quantities is co-efficient of restitution for which a formula must be provided in terms of known functions.

Ghosh and Ray considered the problem of lateral impact at the free end of a cantilever Subsequently Ray applied the same deductions of the problem in case of the load striking at any point of the bar But such application is not valid and leads to error as he assumed the shearing force to be equal in values with the change in the shearing force, incurred in crossing the struck point, and they could not anticipate correct boundary cnditions at the struck point.

Thus, as the force between the load and the beam is not known apriori, these workers had assumed this force as known function of time The theories are not general

The present writer develops the dynamics of the problem in a straight forward way Operational method due to Heaviside has been employed to work out the problem. The main idea upon which the theory is built up is that, the cantilever behaves like a loaded beam so long as the load is in contact with it Further within limits of elasticity and for a finite and constant area of contact, the pressure exerted by the hammer during impact is proportional to the compression of the hammer Displacement of the centre of gravity of the hammer at any instant is the displacement of the cantilever together with compression of the load. The shearing force is not continuous near the struck point The pressure exerted by the hammer is equal to the alteration in the value of the shearing force in the cantilever incurred in crossing the struck point

EXPLANATION OF SYMBOLS

l = length of the cantilever $= a + b$

a = segment of cantilever towards the fixed end

b = segment of cantilever towards the free end

t = variable time

x = variable measured along the length of cantilever, being fixed at $x = 0$, and free at $x = l$

y_a = displacement of the struck point $x = a$

y_1 = displacement at any point $x < a$

y_2 = displacement at any point, $a < x$

M = mass of the cantilever

E_1 = Young's modulus of the material of the cantilever
I = moment of inertia of cross section about the neutral axis
k = radius of gyration of cross section
c = velocity of longitudinal waves in the cantilever
m = mass of the load (hammer)
v_0 = velocity of the load before impact
u = compression of the load
z = displacement of the load $= y_a+u$
$nl = \gamma$, $na = k_1\gamma$, $nb = k_2\gamma$;
E_2 Elastic constant of the load (other than Young's modulus)
$\frac{M}{m}$ = 'mass ratio'
$D = \frac{d}{dt}$ (operator)
ϕ_0 = duration of impact
J =
$n = \sqrt{\frac{iD}{KC}}$
$\frac{E_1 I}{K^2C^2} = \frac{M}{l}$

The equation of motion of the transverse vibration of the bar is given by

$$\frac{d^2y}{dt^2}+k^2c^2\frac{d^4y}{dx^4} = 0 \quad \dots \quad (1)$$

which can be written as

$$\frac{d^4y}{dx^4}+\frac{D^2}{k^2c^2}y = 0 \quad \dots \quad 2)$$

The solution of this equation is given by

$$y = R_1 \sinh nx+R_2 \cosh nx+R_3 \sin nx+R_4 \cos nx$$

where $$n = \sqrt{\frac{iD}{kc}}$$

In our problem the load strikes the cantilever at $x = a$, and if y_a is the displacement of the struck point, we have

$$\text{at} \quad x = 0, \quad y = 0, \quad \text{and} \quad \frac{dy}{dx} = 0, \quad \dots \quad (3.1)$$

$$\text{at} \quad x = l, \quad \frac{d^2y}{dx^2} = 0, \quad \text{and} \quad \frac{d^3y}{dx^3} = 0, \quad \dots \quad (3.2)$$

as also

at $x = a$, $y_1 = y_a = y_2$. (3.3)

$$\left(\frac{dy_1}{dx}\right)_{x=a} = \left(\frac{dy_2}{dx}\right)_{x=a} \tag{3.4}$$

$$\left(\frac{d^2y_1}{dx^2}\right)_{x=a} = \left(\frac{d^2y_2}{dx^2}\right)_{x=a} \tag{3.5}$$

With the help of equations (3), we have

$$y_1 = y_a \frac{\Delta_1 (\sinh nx - \sin nx) + \Delta_2 (\cosh nx - \cos nx)}{\Delta_0} \tag{4.1}$$

$$y_2 = y_a \frac{\Delta_3[\sinh n(l-x) + \sin n(l-x)] + \Delta_4[\cosh n(l-x) + \cos n(l-x)]}{\Delta_0} \quad \ldots \tag{4.2}$$

where

$$\Delta_1 = 2[\sinh nl \sin nb - \cosh nl \cos nb - \sin nl \sinh nb - \cos nl \cosh nb - \cosh na - \cos na] \quad \ldots \tag{5.1}$$

$$\Delta_2 = 2[\sinh nl \cos nb + \sin nl \cosh nb - \cosh nl \sin nb - \cos nl \sinh nb + \sinh na + \sin na] \tag{5.2}$$

$$\Delta_3 = 2[\cosh nl \cos na - \sinh nl \sin na - \sin nl \sinh na - \cos nl \cosh na + \cos nb - \cosh nb] \tag{5.3}$$

$$\Delta_4 = 2[\cosh nl \sin na - \sinh na \cos nl - \sinh nl \cos na + \sin nl \cosh na - \sin nb + \sinh nb] \quad \ldots \tag{5.4}$$

$$\Delta_0 = 4[\cosh na \sin na - \sinh na \cos na + \sinh nb \cos nb - \cosh nb \sin nb + \cosh na \cosh nb \sin nl - \cos na \cos nb \sinh nl] \quad \ldots \tag{5.5}$$

since $nl = \gamma$, we shall henceforward write $na = k_1\gamma$, $nb = k_2\gamma$, as $k_1 = \dfrac{a}{l}$, and $k_2 = \dfrac{b}{l}$.

The pressure exerted by the load is given by

$$P = m\frac{d^2z}{dt^2} = -E_2 u \text{ (by Hooke's law)} \quad \ldots \tag{6.1}$$

and subsequent motion of the load is represented by the equation

$$m\frac{d^2z}{dt^2} = E_1 I \Delta \left(\frac{d^3y}{dx^3}\right)_{x=a} \quad \ldots \tag{6.2}$$

where $\Delta \dfrac{d^3y}{dx^3}$ is the alteration in the values of $\dfrac{d^3y}{dx^3}$ in crossing the struck point.

and $E_1 I\Delta\left(\frac{d^3y}{dx^3}\right)_{x=a}$ can be written from the values of y_1 and y_2 in eqns. (4)

as $$E_1 I y_a n^3 f(D). \qquad \ldots \quad (6.3)$$

where, at $x = a = l$, i.e., for the struck point at the free end,

$$f(D) = \frac{1+\cosh\gamma\cos\gamma}{\sinh\gamma\cos\gamma-\cosh\gamma\sin\gamma} \qquad \ldots \quad (7.1)$$

and at $x = a = l/2$, i.e., for the middle point of the bar being struck point,

$$f(D) = \frac{2(1+\cosh\gamma\cos\gamma)}{\sinh\gamma\cos^2\frac{\gamma}{2}-\cosh^2\frac{\gamma}{2}\sin\gamma} \qquad \ldots \quad (7.2)$$

With the help of eqn. (6.3), eqn. (6.1) and eqn (6.2), can be written as

$$mD^2y_a+mD^2u-E_1 I y_a n^3 f(D) = DJ \qquad \ldots \quad (8.1)$$

$$mD^2y_a+mD^2u+E_2u = DJ \qquad \ldots \quad (8.2)$$

Now solving these equations for y_a and u, we get,

$$\frac{y_a}{v_0} = \frac{F(D)}{F_1(D)} = \frac{D}{D^2-\frac{E_1 I}{m}n^3\left(1+\frac{m}{E_2}D^2\right)f(D)} \qquad \ldots \quad (9.1)$$

$$u = -\frac{E_1 I}{E_2}y_a n^3 f(D) \qquad \ldots \quad (9.2)$$

With the help of Heaviside's expansion theorem

$$\frac{y_a}{v_0} = \frac{F(0)}{F_1(0)}+\sum\frac{F(\alpha_s)}{\alpha_s F'_1(\alpha_s)}e^{\alpha_s t} \qquad \ldots \quad (10)$$

where summation extends over all roots of $D = [\alpha_s]$, ($s = 1, 2, 3, \ldots r$)

Now $F(0) = 0$, putting $D = 0$, and $F_1(0) \neq 0$,

as for $x = a = l$, $F_1(0) = \frac{3E_1 I}{Ml^3}$,

For roots of D from $F_1(D) = 0$, we have $F_1(D) = 0$, when from eqn. (9), for $x = a = l$,

$$\frac{1+\cosh\gamma\cos\gamma}{\cosh\gamma\sin\gamma-\sinh\gamma\cos\gamma} = \frac{\frac{m}{M}\gamma}{1-\frac{E_1 I}{E_2}\cdot\frac{m}{M}\frac{\gamma^4}{l^3}} \qquad \ldots \quad (11.1)$$

for $$x - a = \frac{l}{2},$$

$$\frac{2(1+\cosh\gamma\cos\gamma)}{\cosh^2\frac{\gamma}{2}\sin\gamma - \sinh\gamma\cos^2\frac{\gamma}{2}} = \frac{\frac{m}{M}\gamma}{1 - \frac{E_1 I}{E_2}\cdot\frac{m}{M}\cdot\frac{\gamma^4}{l^3}} \qquad \ldots \quad (11.2)$$

Eqns. (11.1) and (11.2), can be solved graphically by plotting for $x - a = l$,

$$\eta_1 = \frac{1+\cosh\gamma\cos\gamma}{\cosh\gamma\sin\gamma - \sinh\gamma\cos\gamma} \qquad \ldots \quad (11.3)$$

$$\eta_2 = \frac{\frac{m}{M}\gamma}{1 - \frac{E_1 I}{E_2}\cdot\frac{m}{M}\cdot\frac{\gamma^4}{l^3}} \qquad \ldots \quad (11.4)$$

and for $x - a = \frac{l}{2}$,

$$\eta_1 = \frac{2(1+\cosh\gamma\cos\gamma)}{\cosh^2\frac{\gamma}{2}\sin\gamma - \sinh\gamma\cos^2\frac{\gamma}{2}} \qquad \ldots \quad (11.5)$$

and

$$\eta_2 = \frac{\frac{m}{M}\gamma}{1 - \frac{E_1 I}{E_2}\,\frac{m}{M}\,\frac{\gamma^4}{l^3}} \qquad \ldots \quad (11.6)$$

As $\eta_2 - \gamma_s$ curve exists only in the positive direction, the different values of γ_s are obtained from the intersections of two sets of curves $\eta_1 \sim \gamma_s$ and $\eta_2 \sim \gamma$, lying entirely in positive quadrants. Thus γ_s assumes different sets of values for different struck points.

Therefore $\quad nl = \gamma_s,\ \gamma_s =$ pure number (for $s = 1, 2, 3, \ldots r$) $\qquad \ldots \quad (12.1)$

Thus $\quad D - [\alpha_s] = \pm iq_s, \qquad \ldots \quad (12.2)$

where $$q_s = \gamma_s^2 \cdot \sqrt{\frac{E_1 I}{M l^3}}, \qquad \ldots \quad (12.3)$$

After some methematical manipulation and with the help of Heaviside's expansion theorem (eqn, 10) we finally get,

for $x-a=l$, i e., at the free end of the bar

$$y_l=\sum_{s=1}^{r}\frac{4v_0}{\dfrac{1+\dfrac{3m}{E_2}q_s^2}{1-\dfrac{m}{E_2}q_s^2}+\dfrac{M}{m}\left[1-\dfrac{m}{E_2}q_s^2\right]+\dfrac{2\gamma_s}{\coth\gamma_s-\cot\gamma_s}}\times\frac{\sin q_s t}{q_s}. \quad \ldots \quad (13.1)$$

The same expression was obtained by Ghosh—Ray but their deduction was faulty as stated earlier.

Again at $x-a=\frac{l}{2}$, i.e., for centre-struck case,

$$y_{\frac{l}{2}}=4v_0\sum\frac{\sin q_s t/q_s}{\dfrac{1+\dfrac{3m}{E_2}q_s^2}{1-\dfrac{m}{E_2}q_s^2}+\gamma_s\dfrac{\cosh\gamma_s\sin\gamma_s-\sinh\gamma_s\cos r_s}{1+\cosh\gamma_s\cos\gamma_s}} \quad \ldots \quad (13\ 2)$$

$$+\gamma_s\frac{\cosh\gamma_s\cos^2\frac{\gamma_s}{2}-\sinh\gamma_s\sin\gamma_s-\cosh^2\frac{\gamma_s}{2}\cos\gamma_s}{\sinh\gamma_s\cos^2\frac{\gamma_s}{2}-\cosh^2\frac{\gamma_s}{2}\sin\gamma_s} \quad \ldots \quad (13.2)$$

For hard load, since E_2 is taken to be infinity, $3m/E_2$, and m/E_2 is zero Finally the expression for displacement of the cantilever at the specific struck points can be represented in some form as

$$y_a=4v_0\left[\frac{A_1}{q_1}\sin q_1t+\frac{A_2}{q_2}\sin q_2t+\frac{A_3}{q_3}\sin q_3t+\ldots\right].$$

where A_1, A_2, A_3, etc. and q_1, q_2, etc. have their values as required. Different terms of this series represent, different modes of vibration, excited during impact, whose periods are obtained from eqn. (12.3) Similar type of expression for displacement was shown by Prescot in case of a blow at the free end of a cantilever but amplitudes of different modes etc. in his case are however different from those of ours. Only limited number of terms of the series is required to compute the experimental observations.

RESULTS

The values of ϵ', ϵ'', tan δ and τ along with the macroscopic viscosities (η) of all the compounds at different temperatures are given in Tables I–V. The τ-values of chlorobenzene, bromobenzene and α-chloronaphthalene at different temperatures along with those reported by Hennelly *et al.* (1948) has been represented graphically in Figure 2 for the sake of comparison. The values of heats

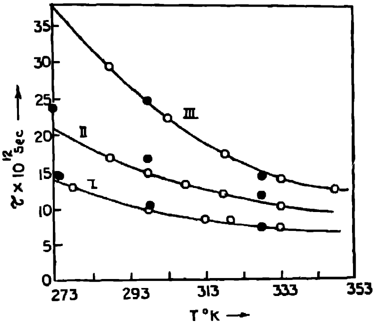


Fig 2 Graphs of τ vs T
Curve I—Chlorobenzene, Curve II—Bromobenzene
Curve III - α-Chloronaphthalene (The scale for τ is to be increased by a factor of two)
Open circles denote the τ-values obtained by the authors
Solid circles are those by Hennelly *et al* (1948)

of activation for dielectric relaxation (ΔH_τ) and viscous flow (ΔH_n) have been obtained respectively from the graphs log ($\tau \cdot T$) against $1/T$ and log η against $1/T$ as usual Some of these curves are shown in Figures 3a, 3b and 3c. The ΔH_τ and ΔH_η- values and their ratios (γ) are given at the foot of the Tables.

TABLE I

Chlorobenzene

Wave length (λ) = 3.14 cm.

Temp °K	ϵ'	ϵ''	tan δ	$\tau \times 10^{12}$ sec	η in millipoise	$\frac{\tau T}{\eta^\gamma} \times 10^7$
278	4.650	1.770	0.3809	12.69	9.55	9.13
298	4.707	1.400	0.2974	9.80	7.56	8.70
313	4.726	1.226	0.2595	8.51	6.38	8.78
320	4.501	1.058	0.2345	8.08	5.96	8.88
333	4.529	0.953	0.2104	7.20	5.26	8.87

ΔH_τ = 1.19K Cal/mole γ = 0.60

ΔH_η = 1.99 K.Cal/mole

TABLE II

Bromobenzene

Wave length (λ) = 3.14 cm

Temp °K	ε'	ε''	tan δ	$\tau \times 10^{12}$ sec	η in millipoise	$\frac{\tau T}{\eta^{\gamma}} \times 10^{7}$
288	3.852	1.454	0.3773	17.03	11.89	9.21
298	3.976	1.378	0.3465	14.87	10.55	9.02
308	4.075	1.302	0.3195	13.19	9.17	9.09
318	4.137	1.234	0.2983	12.05	8.20	9.25
333	4.230	1.110	0.2624	10.28	7.05	9.15

$\Delta H_{\tau} = 1.49$ K Cal/mole $\quad \gamma = 0.68$

$\Delta H_{\eta} = 2.21$ K.Cal/mole

TABLE III

Metadichlorobenzene

Wavelength (λ) = 3.14 cm

Temp °K	ε'	ε''	tan δ	$\tau \times 10^{12}$ sec	η in millipoise	$\frac{\tau T}{\eta^{\gamma}} \times 10^{7}$
288	3.546	1.049	0.2959	15.13	11.01	13.53
303	3.601	0.931	0.2585	12.82	9.17	13.20
318	[illegible]86	0.851	0.2374	11.87	7.80	13.89
333	3.630	0.740	0.2039	9.95	6.67	13.15
348	3.620	0.678	0.1874	9.18	5.82	13.55

$\Delta H_{\tau} = 1.02$ K Cal/mole $\quad \gamma = 0.49$

$\Delta H_{\eta} = 2.09$ K.Cal/mole

IV

1, 2, 4-trichlorobenzene

Wavelength (λ) = 3.14 cm

Temp °K	ε'	ε''	tan δ	$\tau \times 10^{12}$ sec	η in millipoise	$\frac{\tau T}{\eta^{\gamma}} \times 10^{7}$
288	2.900	0.8143	0.2806	30.48	26.04	30.94
303	2.917	0.7563	0.2593	26.03	18.28	31.13
318	2.983	0.7269	0.2437	22.43	13.65	30.91
333	3.026	0.6905	0.2282	20.16	10.84	31.31
348	3.003	0.6384	0.1775	18.41	9.01	31.72

$\Delta H_{\tau} = 1.11$ K.Cal/mole

$\Delta H_{\eta} = 3.47$ K.Cal/mole

TABLE V

α-Chloronaphthalene

Wavelength (λ) = 3 14 cm

Temp °K	ε'	ε''	tan δ	$\tau \times 10^{12}$ sec	η in millipoise	$\frac{\eta T}{\eta^{\gamma}} \times 10^{7}$
288	3 096	1 533	0 4950	59 6	42.66	17 19
303	3.215	1 478	0.4596	44 9	29.17	17 19
318	3 150	1 003	0 3184	34 6	20 30	17.37
333	3 170	0 823	0 2596	27.3	14.69	17 50
348	3.215	0 820	0 2551	24 9	11.22	19 67
	ΔH_{τ} = 2 71 K Cal/mole			γ = 0 61		
	ΔH_{η} —4.42 K.Cal/mole					

TABLE VI

Compound	Semi-axial lengths in Å			f	$4\pi abcf$ in Å^3	$\frac{\tau KT}{3\eta(\eta_o/\eta)^{1-\gamma}}$ in Å^3	
	a	b	c			η_o — 1 mp	η_o — 0.2 mp
C_6H_5Cl	3 46	3 11	1 50	1 44	97.4 (67.6)	40 8	77 9
C_6H_5Br	3 56	3 11	1 70	1.27	100 0 (78.8)	42 1	70.4
m-$C_6H_4Cl_2$	3 31	3 11	1 70	1 35	87 3 (64 7)	61 9	140 7
α-$C_{10}H_7Cl$	4.04	3 46	1 50	1.35	118 6 (87 8)	79.6	149 1

DISCUSSION

a) *Dependence of time of relaxation* (τ) *on the macroscopic viscosity* (η)

The Debye relation as modified by Perrin (1934) for ellipsoidal molecules with rigid dipoles is

$$\tau = \zeta/2KT \qquad \dots (1)$$

where

$$\zeta = 8\pi abcf\eta_{int} \qquad \dots (2)$$

ζ being a measure of the viscous force inhibiting the rotation of the dipole lying along one of the axes of the ellipsoid about the other two axes, a, b, c are the lengths of the semi axes of the ellipsoid, f is a factor tabulated by Budo *et al.* (1939) and η_{int} is the internal friction of the medium in which ellipsoids are rotating. As

Eqn. (1) shows, the product $\tau \cdot T$ is a measure of ζ. The nature of dependence on the shape and size of molecules and on temperatures can be seen from the plots $\tau \cdot T$ against T for all the compounds as shown in Figure 4. It is seen that the values of τT (i.e. ζ) decreases in the order toluene $* <$ chlorobenzene $<$ meta-dichlorobenzene $<$ bromobenzene $< \alpha$-chloronaphthalene. The viscosities η of

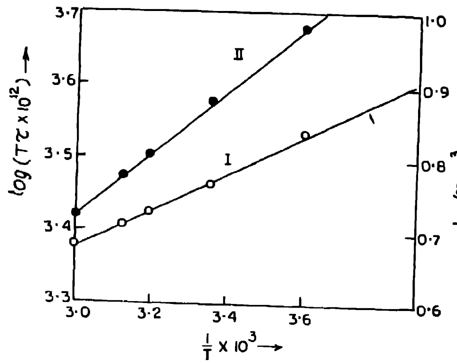


Fig. 3a. Chlorobenzene

Curve I—Plot of log (τ.T) vs 1/T

Curve II—Plot of log η vs 1/T

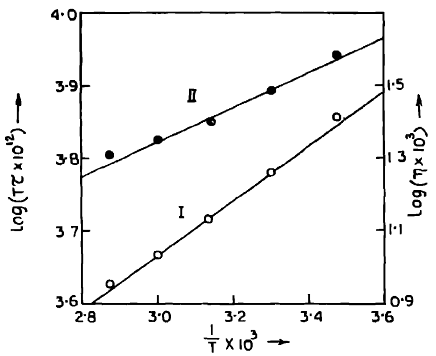


Fig. 3b 1, 2, 4-trichlorobenzene

Curve I—Plot of log (τ T) vs 1/T

Curve II—Plot of log η vs 1/T

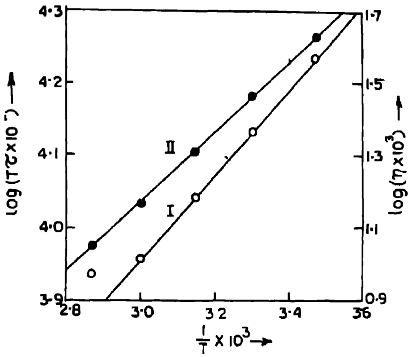


Fig. 3e. α-Chloronaphthalene
Curve I—Plot of log (τ T) vs 1/T
Curve II Plot of log η vs 1/T

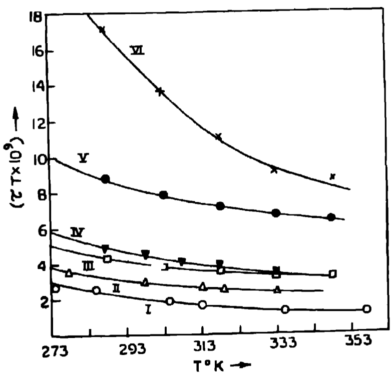


Fig. 4. Graphs showing the variation of τ.T against T
Curve (*i*)—Toluene; Curve (*ii*)—Chlorobenzene;
Curve (*iii*)—m-dichlorobenzene; Curve (*iv*)—Bromobenzene;
Curve (*v*)—1, 2, 4-trichlorobenzene; Curve (*vi*)—α-Chloronaphthalene.

the compounds also decrease in the same order. The first four compounds have their molecular volume almost equal and so the values of ζ in those cases is mainly a function of macroscopic viscosity. The functional relationship between ζ and η can be advantageously studied from the plots of log(τT) *vs* log η. Two of such graphs are shown in Figures 5a and 5b. The linear variation of log (τT) with log η can be designated by a relation $\log(\tau T) = A + \gamma \log \eta$...(3). The slope of the curve gives the value $\gamma = \Delta H_T / \Delta H_\eta$ Substitution of Eqn.(3) in Eqn. (1) gives

$$\zeta = D \,.\, \eta^{\gamma} \qquad \ldots \quad (4)$$

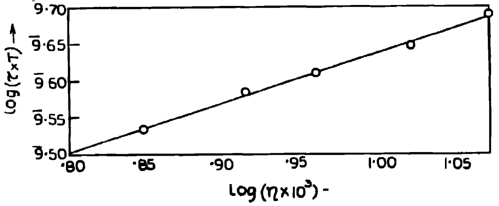


Fig. 5a. Variation of log (τ.T) with log η for Bromobenzene

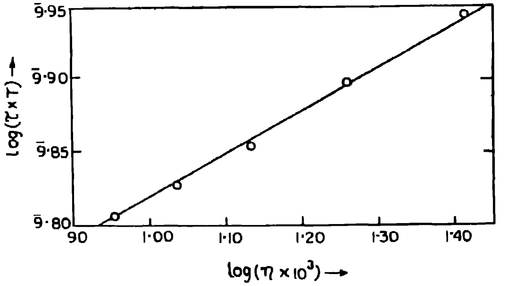


Fig 5b. Variation of log (τT) with log for η 1, 2, 4-trichlorobenzene

which is of the same form as derived in a previous communication (Sinha *et al.* 1965) Following similar arguments as given in that paper ζ may be written

$$\zeta = 2\tau KT = C. \left(\frac{\eta_0}{\eta}\right)^{1-\gamma} . \eta. \qquad \ldots \quad (5)$$

where C has the dimension of a volume, η_0 is a constant having the same dimension as that of η and $\left(\frac{\eta_0}{\eta}\right)^{1-\gamma} \cdot \eta$ is tentatively indentified with η_{int}.

Combining the equations (1), (2) and (5)

$$C = 8\pi abcf \qquad \ldots \quad (6a)$$

*The data for toluene is taken from a previous paper by Bhattacharyya *et al.* (1964).

and

$$\frac{\tau KT}{3\eta.(\eta_0/\eta)^{1-\gamma}} = \frac{4}{3}\pi\, abcf \qquad .. \quad (6b)$$

The value of η_0 can not be determined, but with certain tentative values of η_0 a comparison may be made between the experimentally obtained values of $\tau kT/3\eta(\eta_0/\eta)^{1-\gamma}$ and the calculated values of $4/3\ \pi abcf$ The values of a, b, c are determined from atomic radii (Fischer, 1946) and those of 'f' are taken from the Table of Budo *et al.* (1939). All these data are given in Table VI. The value of molecular volume ($4/3\ \pi abc$) are given in brackets

It is seen from Table VI that if the value of η_0 is properly chosen, some amount of agreement between the experimental values of $\frac{\tau kT}{3\eta\ (\eta_0/\eta)^{1-\gamma}}$ and $\frac{4}{3}\ \pi\ abcf$ may be obtained This may provide some justification for identifying $\eta\ .\ (\eta_0/\eta)^{1-\gamma}$ i e η^{γ} for a certain compound as a measure of internal friction as has been suggested earlier (Sinha *et al.* 1965).

b) *Influence of dipolar interaction on relaxation time*

The τ-values of the molecules of chlorobenzene, bromobenzene, metadichlorobenzene and α-chloronaphthalene (having almost identical dipole moments) in the liquid state have been compared with those of the molecules in dilute solutions in suitable non-polar solvents, whose viscosities are either equal to or greater than those of the pure liquids*. The results are shown graphically in Figures 6a and 6b.

It is seen from the figure 6a that the τ-values of chlorobenzene in the liquid state are slightly smaller than those in solutions in CCl_4 which may presumably be due to the slightly lower values of η in the case of the pure liquid.

In the case of bromobenzene and metadichlorobenzene, however, the τ-values in the pure liquid, are almost equal to those in solution in CCl_4, the viscosities of the pure liquids and the solvent being almost the same. The behaviour of α-chloronaphthalene is different from that of the above three compounds. It can be seen from Figure 6b that the τ-values of α-chloronaphthalene in solution in a very viscous solvent paraffin over the range of temperature investigated are much smaller than those in the pure liquid in the same temperature range, although the viscosity of the pure liquid is much smaller than that of paraffin. Curtiss *et al* (1952) also reported that the value of λ_{max} ($= 2\pi c\tau$) of α-chloronaphthalene in the liquid state at 20°C ($\eta = 33.3$ mp) is 11 cm. which is only slightly smaller than the value of 12 cm. for λ_{max} in solution in nujol ($\eta = 1080$ mp) at the same temperature.

The results in the cases of chloro, bromo- and metadichlorobenzene obtained in the present investigation do not show any decrease in the τ-values in solution

*The data on solutions are taken from the works of Sinha *et al.* (to be published soon).

as compared to those in the respective pure liquids. This can not be reconciled with the conclusion made by Smyth (1955) that the τ-values of these liquids are

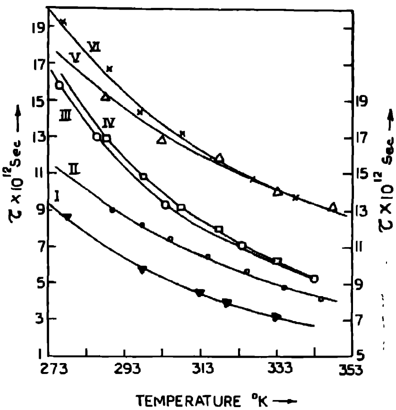


Fig. 6a. Comparison of the τ-values at different temperatures
Curve (*i*) - Chlorobenzene-pure liquid ;
Curve (*ii*)—Chlorobenzene in solution in CCl_4
Curve (*iii*)—Bromobenzene-pure liquid
Curve (*iv*)—Bromobenzene in solution in CCl_4
Curve (*v*)—Metadichlorobenzene-pure liquid
Curve (*vi*)—Metadichlorobenzene in solution in CCl_4 The scale for τ values is given on the right.

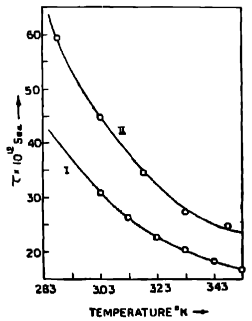
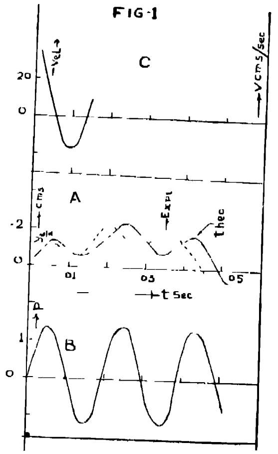


Fig. 6b. Comparison of the τ-values at different temperatures
Curve (*i*)—α-Chloronaphthalene-pure liquid
Curve (*ii*)—α-Chloronaphthalene in solution in paraffin.

For verification of the theory, the following data are taken from a paper of Banerjee (1964). Length of Cantilever, 90 cms dia 1·27 cms., struck point 45 cms (mid-pt), spherical brass load, 'mass ratio' 3.71 (calculated), $v_0 = 88.5$ cms. per sec.



The experimental photographic curve for the struck point is projected upto a squared paper by the help of an epidiascope The curve traced out by the surface of the cantilever during and shortly after impact is drawn upon the paper The displacement-time curve as calculated from the theory with two terms of the series, is then superimposed upon the experimental curve with same units These curves are shown in Fig. 1. They are identical upto the time the contact did not cease, i.e. upto about .011 sec (see the said paper) This clearly upholds the validity of the present theory After this time (·011 sec) also the curves are strikingly similar and very small difference is due to after-impact effects, which has been discussed in Part III of this series.

PRESSURE EXERTED BY THE HAMMER

Substituting values of y_a and u, of equations (9.1) and (9.2) respectively in eqn. (6.1) we write,

$$= -mv_0 \frac{F(D)}{F_2(D)} \qquad \cdots \quad (14.1)$$

where, at $x = a = l$,

$$F_2(D) = 1 + \frac{m}{E_2} D^2 + \gamma \frac{m}{M} \frac{\sinh \gamma \cos \gamma - \cosh \gamma \sin \gamma}{1 + \cosh \gamma \cos \gamma} \quad \ldots \quad (15.1)$$

and at $\quad x = a = \dfrac{l}{2}$,

$$F_2(D) = 1 + \frac{m}{E_2} D^2 + \gamma \frac{m}{M} \frac{\cos^2 \frac{\gamma}{2} \sinh \gamma - \cosh^2 \frac{\gamma}{2} \sin \gamma}{2(1 + \cosh \gamma \cos \gamma)} \quad \ldots \quad (15.2)$$

For roots of D from $F_2(D) = 0$, we find that $D = [\alpha_s], = \pm i q_s$, has same roots as obtained by equations (11), for respective struck points

With the help of Heaviside's expansion theorem

$$P = -m v_0 \left[\frac{F(0)}{F_2(0)} + \sum_{s=1}^{r} \frac{F(\alpha_s)}{\alpha_s \cdot F'_2(\alpha_s)} e^{\alpha_s t} \right] \quad \ldots \quad (16)$$

Here again $F(0) = 0$ and $F_2(0) \neq 0$, putting $D = 0$

Finally after some mathematical manipulation and simplification we get,

Pressure at $x = a = l$,

$$P_l = -4 m v_0 \sum_{s=1}^{r} \frac{q_s \sin q_s t}{1 + \frac{3m}{E_2} q_s^2 + \frac{M}{m}\left[1 - \frac{m}{E_2} q_s^2\right] + 2\gamma_s \left[1 - \frac{m}{E_2} q_s^2\right] \frac{1}{\coth \gamma_s - \cot \gamma_s}} \quad \ldots \quad (17.1)$$

This expression is same as obtained by Ghosh and Ray

and Pressure at $x = a = \dfrac{l}{2}$,

$$P_{\frac{l}{2}} = -4 m v_0 \sum \frac{q_s \sin q_s t}{1 + \frac{3m}{E_2} q_s^2 + \gamma_s \left[1 - \frac{m}{E_2} q_s^2\right] \frac{\cosh \gamma_s \sin \gamma_s - \sinh \gamma_s \cos \gamma_s}{1 + \cosh \gamma_s \cos \gamma_s} + \frac{m}{M} \gamma_s^2 \frac{\sinh \gamma_s \sin \gamma_s + \cosh^2 \frac{\gamma_s}{2} \cos \gamma_s - \cos^2 \frac{\gamma_s}{2} \cosh \gamma_s}{2(1 + \cosh \gamma_s \cos \gamma_s)}} \quad \ldots \quad (17.2)$$

Further velocity of the bar, which is given by $\dfrac{dy}{dt}$ can be represented in the form

$$v_t = 4 v_0 [A_1 \cos q_1 t + A_2 \cos q_2 t + A_3 \cos q_3 t + \ldots\ldots] \quad \ldots \quad (18)$$

Thus it is possible to directly obtain the velocity of the load from eqn. 18.

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